Zoom Class 6/5/2020

Substitution in a Definite Integral

In Integration of $f(x)$ by substitution method we assume $x = \phi(t)$

and ~~$\int f(x) = \int f(\phi(t)$~~

$$\int f(x) = \int f(\phi(t))\, \phi'(t)\, dt = \int h(t)\, dt$$

$$= \psi(t) \quad \text{——— (1)}$$

In Definite Integration instead of tracking back from t to x we express $\psi(t)$ in terms of 'x'.

i.e., $\psi(t) = F(x)$

$$\Rightarrow \psi(t) = F[\phi(t)] \quad \text{——— (2)}$$

Now, we find values of α, β of 't' corresponding to the values of a, b of x so that $a = \phi(\alpha)$, $b = \phi(\beta)$

$$\Rightarrow \int_a^b f(x)\, dx = F(b) - F(a)$$

$$= F\{\phi(\beta)\} - F\{\phi(\alpha)\}$$

$$= \psi(\beta) - \psi(\alpha)$$

$$= \int_\alpha^\beta h(t)\, dt.$$

Ques} Evaluate $\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}}$

Soln Let $a^2-x^2=z \Rightarrow -2x\,dx = dz \Rightarrow x\,dx = -\frac{1}{2}dz$

also when $x=0$, $z=a^2$,

and $x=a$, $z=0$

$\therefore\ I = -\frac{1}{2}\int_{a^2}^{0} \frac{dz}{\sqrt{z}} = -\frac{1}{2}\int_{a^2}^{0} (z)^{-1/2}\,dz$

$= -\frac{1}{\cancel{2}}\left[\cancel{2}\,z^{1/2}\right]_{a^2}^{0}$

$= -\left[\cancel{a} - (a^2)^{1/2}\right] = -[0-a]$

$= a$

Ans

Ques} Evaluate $\int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3\cos x}\,dx$

Soln $I = \int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3\cos x}\,dx$

$= \int_0^{\pi/4} \frac{\sin x}{4\cos^3 x}\,dx \longrightarrow \left\{\because 4\cos^3 x = 3\cos x + \cos 3x\right\}$

$= \frac{1}{4}\int_0^{\pi/4} \sec^2 x \tan x\,dx$

Put $\tan x = z \Rightarrow \sec^2 x\,dx = dz$

If $x=0 \Rightarrow z = \tan 0 = 0$

If $x=\pi/4 \Rightarrow z = \tan \pi/4 = 1$

$$\Rightarrow I = \frac{1}{4}\int_0^1 z\,dz$$

$$= \frac{1}{8}\left[z^2\right]_0^1$$

$$= \frac{1}{8}[1-0] = \frac{1}{8} \text{ Ans}$$

Ques Evaluate $\displaystyle\int_0^a \frac{x^2}{(a^2-x^2)^{1/2}}\,dx$

Soln let $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

Also when $x = a$, $\theta = \pi/2$

when $x = 0$, $\theta = 0$

$$\Rightarrow I = \int_0^{\pi/2} \frac{a^2\sin^2\theta \cdot a\cos\theta\,d\theta}{(a^2 - a^2\sin^2\theta)^{1/2}}$$

$$= \int_0^{\pi/2} \frac{\cancel{a^3}^{\,2}\sin^2\theta\cos\theta\,d\theta}{\cancel{a}(1-\sin^2\theta)^{1/2}}$$

$$= \int_0^{\pi/2} \frac{a^2\sin^2\theta\cos\theta\,d\theta}{(\cos^2\theta)^{1/2}}$$

$$= a^2\int_0^{\pi/2} \frac{\sin^2\theta\,\cancel{\cos\theta}}{\cancel{\cos\theta}}\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 2\sin^2\theta\,d\theta$$

$$\Rightarrow I = \frac{a^2}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left\{\left[\frac{\pi}{2} - \frac{\sin\pi}{2}\right] - \left[0 - \frac{\sin 0}{2}\right]\right\}$$

$$= \frac{a^2}{2}\left\{\frac{\pi}{2} - 0 - 0 + 0\right\}$$

$$= \frac{a^2\pi}{4} \quad \underline{\text{Ans}}$$

Ques Evaluate $\displaystyle\int_0^1 \frac{dx}{e^x + e^{-x}}$

Soln $\displaystyle I = \int_0^1 \frac{dx}{e^x + e^{-x}}$

$$= \int_0^1 \frac{e^x\,dx}{e^{2x}+1}$$

Put $e^x = z \Rightarrow e^x\,dx = dz$

$$\boxed{\begin{array}{l}\text{if } x=0 \Rightarrow z=1 \\ \text{If } x=1 \Rightarrow z=e\end{array}}$$

$$\Rightarrow I = \int \frac{dz}{1+z^2} = \tan^{-1} z$$

$$\Rightarrow I = \tan^{-1}(e^x)$$

$$= \left[\tan^{-1}(e^x)\right]_0^1 = \tan^{-1}(e) - \tan^{-1}(1)$$

$$= \tan^{-1}e - \frac{\pi}{4} \quad \underline{\text{Ans}}$$